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Scale-invariant universal crossing probability in one-dimensional diffusion-limited coalescence

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Abstract

The crossing probability in the time direction, π_t, is defined for an off-equilibrium reaction–diffusion system as the probability that the system of size L is still active at time t, in the finite-size scaling limit. Exact results are obtained for the diffusion-limited coalescence problem in $1+1$ dimensions with periodic and free boundary conditions using empty interval methods. π_t is a scale-invariant universal function of an effective aspect ratio, L^2/Dt, which is the natural scaling variable for this strongly anisotropic system.

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1. Introduction

The study of crossing probabilities for standard percolation has been the subject of much interest during the last decade [1–16] (for a recent introductory review see [17]).

In two dimensions, the crossing probability may be defined as the probability π to have at least one cluster joining two opposite edges of a rectangular-shaped finite system with length $L_\parallel$ and width $L_\perp$. It turns out that, at the percolation threshold, in the finite-size scaling limit ($L_\parallel \to \infty$, $L_\perp \to \infty$, with $r = L_\perp/L_\parallel$ fixed), the crossing probability is a *scale-invariant universal function*, $\pi(r)$, of the aspect ratio r [1, 2].

Following the numerical work of Langlands *et al* [1], Cardy [2] was able to derive an exact expression for $\pi(r)$. Using the relation between percolation and the q-state Potts model in the limit $q \to 1$ [18] and boundary conformal field theory, he obtained the crossing probability between two non-overlapping segments on the edge of the half-plane at criticality. The corresponding result in the rectangular geometry was then obtained through a conformal mapping. The non-trivial *scale invariance* of $\pi(r)$ is linked to the vanishing of the scaling dimension $x(q)$ of a boundary condition changing operator of the Potts model in the percolation limit, $q \to 1$. Let us note that some of these results have been rigorously proved recently [10, 11]. A related problem concerns the number of incipient spanning clusters at criticality

[12–16]. Exact formulae have also been obtained in this field through conformal and Coulomb-gas methods [13, 14].

In a recent work [19], the critical crossing probability was studied numerically for a strongly anisotropic system, namely, directed percolation in 1 + 1 dimensions. In this case, the crossing probability in the time direction π_t is also the probability that the system of size L remains active at time t. Anisotropic scaling [20, 21] then implies that the appropriate aspect ratio is $r = L^z/t$, where z is the dynamical exponent. Here, too, it was found that, in the finite-size scaling limit, the critical crossing probability is a scale-invariant universal function of an effective aspect ratio which is the product of r by a non-universal constant.

In the present work we continue the examination of critical crossing probabilities in strongly anisotropic systems by considering the case of diffusion-limited coalescence (DLC). This is one of the many actively studied off-equilibrium systems [22–26] which yields itself to an exact analysis [27–40]. We study the problem with periodic and free boundary conditions using the empty interval method, or interparticle distribution function method, which is reviewed in [36, 37].

The case of periodic boundary conditions is studied in section 2 using the standard empty interval method for a finite discrete system [33–35]. In section 3 we use a modification of the standard method to treat the problem with free boundary conditions. The results are discussed in section 4.

2. Diffusion-limited coalescence with periodic boundary conditions

We consider the time evolution of DLC on a one-dimensional lattice with L sites and periodic boundary conditions. Each site is in one of two states, either vacant or occupied by a particle A. The dynamics is governed by the following processes

$$A\varnothing \overset{D}{\longleftrightarrow} \varnothing A \quad \text{(diffusion)} \qquad AA \overset{D}{\longrightarrow} \begin{cases} A\varnothing \\ \varnothing A \end{cases} \quad \text{(coagulation)} \tag{1}$$

with the *same* rate D. When a particle jumps with rate D on a nearest-neighbour site which is already occupied, the two particles coalesce *immediately* on this site. Thus coagulation may occur either to the left or to the right. To simplify we assume that the L sites are occupied with probability one in the initial state at $t = 0$. As a consequence, the probability distribution of the particles A is translation invariant at later time $t \geqslant 0$.

We study the time evolution of the system using the empty interval method [29]. Let the symbol $\bullet$ ($\circ$) denote an occupied (vacant) site. The probability for a given interval of length n to be empty at time t,

$$I_n(t) = \mathrm{Prob}(\overbrace{\circ\circ\cdots\circ\circ}^{n}) \tag{2}$$

is translation invariant on the periodic system with uniform initial conditions. Its time evolution involves the probability

$$F_n(t) = \mathrm{Prob}(\bullet\overbrace{\circ\circ\cdots\circ\circ}^{n}) = \mathrm{Prob}(\overbrace{\circ\circ\cdots\circ\circ}^{n}\bullet) \tag{3}$$

to have an empty interval of length n, either preceded or followed by an occupied site. Since

$$\mathrm{Prob}(\overbrace{\circ\circ\cdots\circ\circ}^{n}\bullet) + \mathrm{Prob}(\overbrace{\circ\circ\cdots\circ\circ\circ}^{n+1}) = \mathrm{Prob}(\overbrace{\circ\circ\cdots\circ\circ}^{n}) \tag{4}$$

the following relation is obtained:

$$F_n(t) = I_n(t) - I_{n+1}(t). \tag{5}$$

For $n = 1, L-1$, the empty interval probability satisfies the master equation

$$\frac{\mathrm{d}I_n(t)}{\mathrm{d}t} = 2D[F_{n-1}(t) - F_n(t)] = 2D[I_{n-1}(t) - 2I_n(t) + I_{n+1}(t)]. \tag{6}$$

The gain terms correspond to processes in which a particle occupying the first site on the right (left) of an empty interval of length $n-1$ jumps to the right (left) to diffuse or coalesce on the next site, thus leaving behind an empty interval of length n. The loss terms correspond to processes in which a nearby particle enters an empty interval of length n either from the left or from the right. The final form of the difference equation follows from (5).

Equation (6) has to be solved with the boundary conditions

$$I_0(t) = 1 \qquad I_L(t) = 0. \tag{7}$$

The first one results from the expression of the site occupation probability, $F_0(t) = 1 - I_1(t)$, the second follows from the fact that an initially non-empty system remains so at later time, since the coalescence process leaves at least one surviving particle. With a full lattice at $t = 0$, the initial condition corresponds to

$$I_n(0) = \delta_{n,0}. \tag{8}$$

The master equation (6) is solved through the ansatz

$$I_n(t) = \sum_q \phi_q(n)\,\mathrm{e}^{-\omega_q t} \tag{9}$$

where $\phi_q(n) = u_q \sin(qn) + v_q \cos(qn)$ when $\omega_q = 8D\sin^2(q/2)$ is non-vanishing. It takes the form $\phi_0(n) = an + b$ for the zero mode, $\omega_0 = 0$, corresponding to the stationary state. The first boundary condition in (7),

$$I_0(t) = 1 = b + \sum_{q\neq 0} v_q\,\mathrm{e}^{-\omega_q t} \tag{10}$$

leads to $b = 1$ and $v_q = 0$ whereas the second

$$I_L(t) = 0 = 1 + aL + \sum_{q\neq 0} u_q \sin(qL)\,\mathrm{e}^{-\omega_q t} \tag{11}$$

gives $a = -1/L$ and $\sin(qL) = 0$. Thus the empty interval probability can be written as [35]

$$I_n(t) = 1 - \frac{n}{L} + \sum_{k=1}^{L-1} c_k \sin\left(\frac{nk\pi}{L}\right) \exp\left[-8Dt\sin^2\left(\frac{k\pi}{2L}\right)\right]. \tag{12}$$

The stationary state solution $I_n(\infty) = 1 - n/L$ corresponds to a single particle diffusing on the L sites so that, a site being occupied with probability $1/L$, an interval of n sites is non-empty with probability n/L.

According to (8), at $t = 0$ we have

$$\sum_{k=1}^{L-1} c_k \sin\left(\frac{nk\pi}{L}\right) = \frac{n}{L} - 1 \qquad n \neq 0. \tag{13}$$

Making use of the orthogonality relation for the sines,

$$\sum_{n=1}^{L-1} \sin\left(\frac{nk\pi}{L}\right) \sin\left(\frac{nl\pi}{L}\right) = \frac{L}{2}\delta_{k,l} \qquad (k, l = 1, L-1) \tag{14}$$

in equation (13), we obtain

$$\frac{L}{2}c_k = \frac{S_1(k)}{L} - S_2(k) \tag{15}$$

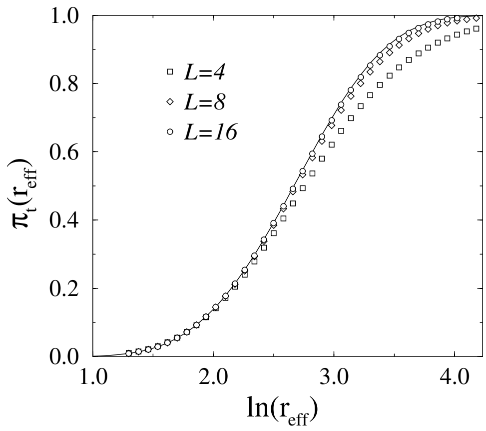

Figure 1. Variation of the scale-invariant crossing probability with the effective aspect ratio for periodic boundary conditions. The solid line corresponds to the asymptotic expression in equation (19).

where

$$\begin{aligned} S_1(k) &= \sum_{n=1}^{L-1} n \sin\left(\frac{nk\pi}{L}\right) = (-1)^{k+1}\frac{L}{2}\cot\left(\frac{k\pi}{2L}\right) \\ S_2(k) &= \sum_{n=1}^{L-1} \sin\left(\frac{nk\pi}{L}\right) = \frac{1-(-1)^k}{2}\cot\left(\frac{k\pi}{2L}\right) \end{aligned} \tag{16}$$

so that finally

$$I_n(t) = 1 - \frac{n}{L} - \frac{1}{L}\sum_{k=1}^{L-1}\cot\left(\frac{k\pi}{2L}\right)\sin\left(\frac{nk\pi}{L}\right)\exp\left[-8Dt\sin^2\left(\frac{k\pi}{2L}\right)\right]. \tag{17}$$

The mean number of particles per site (or site occupation probability), $\rho(t) = 1 - I_1(t)$, has the well-known $t^{-1/2}$ long-time behaviour in the infinite system [27, 28]. Here we are interested in the behaviour of the crossing probability in a system with aspect ratio $r = L^z/t$ with a dynamical exponent $z = 2$ for DLC. The crossing probability in the time direction, $P_t(L, t)$, is the probability that the system of size L is still active at time t.

The probability that the system is in the stationary state, with the last particle on a given site, is equal to the probability $I_{L-1}(t)$ that the $L - 1$ other sites are empty. Since there are L possible choices for the occupied site, one obtains

$$P_t(L, t) = 1 - LI_{L-1}(t) = 2\sum_{k=1}^{L-1}(-1)^{k+1}\cos^2\left(\frac{k\pi}{2L}\right)\exp\left[-8Dt\sin^2\left(\frac{k\pi}{2L}\right)\right]. \tag{18}$$

In the finite-size scaling limit, this leads to the scale-invariant expression

$$\pi_t(r_{\rm eff}) = \lim_{\substack{L,t\to\infty \\ r\,{\rm fixed}}} P_t(L, t) = 2\sum_{k=1}^{\infty}(-1)^{k+1}\exp\left(-\frac{2k^2\pi^2}{r_{\rm eff}}\right) + {\rm O}(L^{-2}) \tag{19}$$

where the crossing probability π_t shown in figure 1 is a universal function of the effective aspect ratio $r_{\rm eff} = r/D = L^2/Dt$.

3. Diffusion-limited coalescence with free boundary conditions

3.1. Master equations for the empty interval probabilities

With free boundary conditions a modified version of the empty interval method is needed to calculate the crossing probability. We define

$$J_{m,n}(t) = \mathrm{Prob}(\overset{1}{\circ}\circ\cdots\overset{m}{\circ}\quad\overset{n+1}{\circ}\cdots\circ\overset{L}{\circ}) \tag{20}$$

as the probability to have two disconnected empty intervals with sites 1 to m and $n+1$ to L empty. Its time evolution depends on the probabilities

$$\begin{aligned} G_{m,n}(t) &= \mathrm{Prob}(\overset{1}{\circ}\circ\cdots\circ\overset{m+1}{\bullet}\quad\overset{n+1}{\circ}\cdots\circ\overset{L}{\circ}) \\ H_{m,n}(t) &= \mathrm{Prob}(\overset{1}{\circ}\circ\cdots\overset{m}{\circ}\quad\overset{n}{\bullet}\circ\cdots\circ\overset{L}{\circ}). \end{aligned} \tag{21}$$

The different probabilities satisfy the relations

$$\begin{aligned} &\mathrm{Prob}(\overset{1}{\circ}\circ\cdots\overset{m}{\circ}\quad\overset{n+1}{\circ}\cdots\circ\overset{L}{\circ}) \\ &\quad = \mathrm{Prob}(\overset{1}{\circ}\circ\cdots\circ\overset{m+1}{\bullet}\quad\overset{n+1}{\circ}\cdots\circ\overset{L}{\circ}) + \mathrm{Prob}(\overset{1}{\circ}\circ\cdots\circ\overset{m+1}{\circ}\quad\overset{n+1}{\circ}\cdots\circ\overset{L}{\circ}) \\ &\quad = \mathrm{Prob}(\overset{1}{\circ}\circ\cdots\overset{m}{\circ}\quad\overset{n}{\bullet}\circ\cdots\circ\overset{L}{\circ}) + \mathrm{Prob}(\overset{1}{\circ}\circ\cdots\overset{m}{\circ}\quad\overset{n}{\circ}\circ\cdots\circ\overset{L}{\circ}) \end{aligned} \tag{22}$$

so that we have

$$G_{m,n}(t) = J_{m,n}(t) - J_{m+1,n}(t) \qquad H_{m,n}(t) = J_{m,n}(t) - J_{m,n-1}(t). \tag{23}$$

As before we assume that the L sites are occupied in the initial state. Thus the system contains at least one particle at later time. The condition of non-emptiness can be written as

$$J_{m,m}(t) = 0 \qquad m = 0, L. \tag{24}$$

One may note that according to the definitions given in (21), $G_{m,m+1}(t) = H_{m,m+1}(t) = J_{m,m+1}(t)$, in agreement with equations (23) and (24).

When $0 < m < n < L$, the empty interval is indeed built of two disconnected parts and its probability satisfies the master equation

$$\begin{aligned} \frac{\mathrm{d}J_{m,n}(t)}{\mathrm{d}t} &= D[G_{m-1,n}(t) + H_{m,n+1}(t) - G_{m,n}(t) - H_{m,n}(t)] \\ &= D[J_{m-1,n}(t) - 2J_{m,n}(t) + J_{m+1,n}(t) + J_{m,n-1}(t) - 2J_{m,n}(t) + J_{m,n+1}(t)] \end{aligned} \tag{25}$$

where the gain terms correspond either to a particle at m jumping to the right or a particle at $n+1$ jumping to the left and the loss terms either to a particle at $m+1$ jumping to the left or a particle at n jumping to the right. When $m = 0$, there is a single empty interval from site $n+1$ to site L and the master equation reads

$$\frac{\mathrm{d}J_{0,n}(t)}{\mathrm{d}t} = D[H_{0,n+1}(t) - H_{0,n}(t)] = D[J_{0,n-1}(t) - 2J_{0,n}(t) + J_{0,n+1}(t)]. \tag{26}$$

In the same way, when $n = L$, we are left with a single empty interval from site 1 to site m and the corresponding probability evolves according to

$$\frac{\mathrm{d}J_{m,L}(t)}{\mathrm{d}t} = D[G_{m-1,L}(t) - G_{m,L}(t)] = D[J_{m-1,L}(t) - 2J_{m,L}(t) + J_{m+1,L}(t)]. \tag{27}$$

Equations (26) and (27) contain the same gain and loss terms as for the corresponding empty intervals in equation (25). They remain valid for $n = L-1$ and $m = 1$, respectively, provided $J_{m,n}(t)$ satisfies the boundary condition

$$J_{0,L}(t) = 1. \tag{28}$$

3.2. Solution of the eigenvalue problem

Looking for the solutions under the form

$$J_{m,n}(t) = \sum_{\omega} \phi_\omega(m,n)\, \mathrm{e}^{-\omega t} \tag{29}$$

the master equations (25)–(27) lead to an eigenvalue problem which has been discussed in detail in [34].

Since $0 \leqslant m < n \leqslant L$, there is a total of $L(L+1)/2$ modes. According to (24), $\phi_\omega(m,n)$ is an antisymmetric combination of eigenfunctions of the second difference operators involved in (25)–(27). The problem is invariant under space reflection so that $\phi_\omega(m,n)$ can be chosen as an eigenfunction of the space reflection operator

$$P : (m,n) \mapsto (L-n, L-m). \tag{30}$$

Three types of solutions are obtained [34]:

- The stationary solution

$$J_{m,n}(\infty) = \phi_0(m,n) = \frac{n-m}{L} \tag{31}$$

which is an eigenstate of P with eigenvalue +1. Its expression follows from the fact that the zero-mode eigenfunction is linear in m and n and has to satisfy the boundary conditions (24) and (28). The first condition leads to the form $\phi_0(m,n) = c(n-m)$ and the second gives $c = 1/L$. It has also a simple physical interpretation: since a single particle remains in the stationary state, a site is occupied with probability $1/L$. Thus the probability to have $L-n+m$ empty sites, from 1 to m and from $n+1$ to L, is given by $1-(L-n+m)/L$.

One may note that with $J_{0,L}(\infty) = 1$, the time-dependent part of $J_{m,n}(t)$ has to satisfy the boundary condition

$$J_{0,L}(t) - J_{0,L}(\infty) = 0 \tag{32}$$

according to (28).

- The $2(L-1)$ one-fermion excitations

$$\begin{aligned} \phi_k^+(m,n) &= \frac{1}{\sqrt{L}}\left[\sin\left(\frac{mk\pi}{L}\right) - \sin\left(\frac{nk\pi}{L}\right)\right] \\ \phi_k^-(m,n) &= \frac{1}{\sqrt{L}}\left[\left(1-\frac{2n}{L}\right)\sin\left(\frac{mk\pi}{L}\right) - \left(1-\frac{2m}{L}\right)\sin\left(\frac{nk\pi}{L}\right)\right] \end{aligned} \tag{33}$$

with $k = 1, L-1$. These functions are antisymmetric eigenstates of P such that $P\phi_k^\pm = \pm(-1)^k\phi_k^\pm$. They vanish when $m = 0$ and $n = L$ in agreement with (32). The excitation energies are given by

$$\omega_k = 4D\sin^2\left(\frac{k\pi}{2L}\right). \tag{34}$$

Actually the odd eigenstate of P, $\phi_k^-(m,n)$, is the combination of a one-fermion excitation with a zero mode.

- The $(L-1)(L-2)/2$ two-fermion excitations

$$\phi_{kl}(m,n) = \frac{2}{L}\left[\sin\left(\frac{mk\pi}{L}\right)\sin\left(\frac{nl\pi}{L}\right) - \sin\left(\frac{ml\pi}{L}\right)\sin\left(\frac{nk\pi}{L}\right)\right] \tag{35}$$

with $1 \leqslant k < l \leqslant L-1$. These antisymmetric two-particle states are eigenstates of P with eigenvalues $(-1)^{k+l+1}$ and they satisfy the boundary condition (32). The corresponding eigenvalues read

$$\omega_{kl} = \omega_k + \omega_l = 4D\left[\sin^2\left(\frac{k\pi}{2L}\right) + \sin^2\left(\frac{l\pi}{2L}\right)\right]. \tag{36}$$

The solution satisfying the boundary conditions can be written as the expansion

$$J_{m,n}(t) = \frac{n-m}{L} + \sum_{k=1}^{L-1}\left[\sum_{\alpha=\pm} a_k^\alpha \phi_k^\alpha(m,n)\right] \mathrm{e}^{-\omega_k t} + \sum_{k=1}^{L-2}\sum_{l=k+1}^{L-1} b_{kl}\phi_{kl}(m,n)\,\mathrm{e}^{-\omega_{kl}t}. \tag{37}$$

All the sites are occupied with probability one in the initial state, so that

$$J_{m,n}(0) = \delta_{m,0}\delta_{n,L}. \tag{38}$$

Thus, for $0 < m < n < L$, we have

$$-\phi_0(m,n) = \sum_{k=1}^{L-1}\sum_{\alpha=\pm} a_k^\alpha \phi_k^\alpha(m,n) + \sum_{k=1}^{L-2}\sum_{l=k+1}^{L-1} b_{kl}\phi_{kl}(m,n). \tag{39}$$

The coefficients of the eigenvalue expansion can be determined by making use of the orthogonality relations between the different eigenfunctions. Following [34], let us define surface and bulk scalar products for arbitrary functions f and g as

$$\begin{aligned}
\langle f|g\rangle_{\mathrm{s}} &= \sum_{m=1}^{L-1} f(m,L)g(m,L) + \sum_{n=1}^{L-1} f(0,n)g(0,n) \\
\langle f|g\rangle_{\mathrm{b}} &= \sum_{n=2}^{L-1}\sum_{m=1}^{n-1} f(m,n)g(m,n).
\end{aligned} \tag{40}$$

It turns out that the one-fermion eigenfunctions in (33) are orthogonal for the surface scalar product whereas the two-fermion eigenfunctions in (35) are orthogonal for the bulk one:

$$\begin{aligned}
&\left\langle\phi_k^\alpha\middle|\phi_{k'}^\beta\right\rangle_{\mathrm{s}} = \delta_{kk'}\delta_{\alpha\beta} &&(k,k' = 1, L-1;\ \alpha,\beta = \pm) \\
&\langle\phi_{kl}|\phi_{k'l'}\rangle_{\mathrm{b}} = \delta_{kk'}\delta_{ll'} &&(0 < k < l < L;\ 0 < k' < l' < L).
\end{aligned} \tag{41}$$

Thus, taking appropriate scalar products in (39), one obtains

$$\begin{aligned}
a_k^+ &= -\left\langle\phi_k^+\middle|\phi_0\right\rangle_{\mathrm{s}} = \frac{2S_1(k)}{L^{3/2}} - \frac{S_2(k)}{L^{1/2}} = -\frac{1+(-1)^k}{2\sqrt{L}}\cot\left(\frac{k\pi}{2L}\right) \\
a_k^- &= -\langle\phi_k^-|\phi_0\rangle_{\mathrm{s}} = \frac{S_2(k)}{\sqrt{L}} = \frac{1-(-1)^k}{2\sqrt{L}}\cot\left(\frac{k\pi}{2L}\right) \\
b_{kl} &= -\langle\phi_{kl}|\phi_0\rangle_{\mathrm{b}} - \sum_{k'=1}^{L-1}\sum_{\alpha=\pm} a_{k'}^\alpha\left\langle\phi_{kl}\middle|\phi_{k'}^\alpha\right\rangle_{\mathrm{b}} = \langle\phi_{kl}|\phi_0\rangle_{\mathrm{b}} \\
&= \frac{(-1)^k - (-1)^l}{2L}\cot\left(\frac{k\pi}{2L}\right)\cot\left(\frac{l\pi}{2L}\right).
\end{aligned} \tag{42}$$

The relation

$$\sum_{k'=1}^{L-1}\phi_{k'}^\pm(m,n)\left\langle\phi_{k'}^\pm\middle|\phi_0\right\rangle_{\mathrm{s}} = \phi_0(m,n) \tag{43}$$

has been used in the calculation of b_{kl}. The final form of $J_{m,n}(t)$ satisfying the initial and boundary conditions follows from (37) and (42).

3.3. Crossing probability

Since $J_{n-1,n}(t)$ gives the probability that all the sites are empty, except site n which is occupied by the last particle, the probability that the system is still active at time t is given by

$$\begin{aligned} P_t(L,t) &= 1 - \sum_{n=1}^{L} J_{n-1,n}(t) = -\sum_{k=1}^{L-1}\left[\sum_{\alpha=\pm} a_k^\alpha \sum_{n=1}^{L} \phi_k^\alpha(n-1,n)\right] \mathrm{e}^{-\omega_k t} \\ &\quad - \sum_{k=1}^{L-2}\sum_{l=k+1}^{L-1} b_{kl}\left[\sum_{n=1}^{L}\phi_{kl}(n-1,n)\right]\mathrm{e}^{-\omega_{kl}t}. \end{aligned} \tag{44}$$

Straightforward but lengthy calculations lead to the following results for the different sums over n:

$$\begin{aligned} &\sum_{n=1}^{L}\phi_k^+(n-1,n) = 0 \\ &\sum_{n=1}^{L}\phi_k^-(n-1,n) = -2\frac{1-(-1)^k}{L^{3/2}}\cot\left(\frac{k\pi}{2L}\right) \\ &\sum_{n=1}^{L}\phi_{kl}(n-1,n) = \frac{1-(-1)^{k+l}}{L}\frac{\sin\left(\frac{k\pi}{L}\right)\sin\left(\frac{l\pi}{L}\right)}{\sin\left[\frac{(k-l)\pi}{2L}\right]\sin\left[\frac{(k+l)\pi}{2L}\right]}. \end{aligned} \tag{45}$$

Thus we obtain

$$\begin{aligned} P_t(L,t) &= \frac{4}{L^2}\sum_{\substack{k=1\\ k\ \mathrm{odd}}}^{L-1}\cot^2\left(\frac{k\pi}{2L}\right)\cot^2\left(\frac{l\pi}{2L}\right)\exp\left[-4Dt\sin^2\left(\frac{k\pi}{2L}\right)\right] \\ &\quad - \frac{8}{L^2}\sum_{k=1}^{L-2}\sum_{\substack{l=k+1\\ k+l\ \mathrm{odd}}}^{L-1}(-1)^k\frac{\cos^2\left(\frac{k\pi}{2L}\right)\cos^2\left(\frac{l\pi}{2L}\right)}{\sin\left[\frac{(k-l)\pi}{2L}\right]\sin\left[\frac{(k+l)\pi}{2L}\right]} \\ &\quad \times\exp\left\{-4Dt\left[\sin^2\left(\frac{k\pi}{2L}\right)+\sin^2\left(\frac{l\pi}{2L}\right)\right]\right\}. \end{aligned} \tag{46}$$

In the finite-size scaling limit, the crossing probability displays the scale-invariant dependence on the effective aspect ratio $r_{\mathrm{eff}} = L^2/Dt$ and reads

$$\begin{aligned} \pi_t(r_{\mathrm{eff}}) &= \lim_{\substack{L,t\to\infty\\ r\ \mathrm{fixed}}} P_t(L,t) \\ &= \frac{16}{\pi^2}\sum_{\substack{k=1\\ k\ \mathrm{odd}}}^{\infty}\frac{1}{k^2}\exp\left(-\frac{k^2\pi^2}{r_{\mathrm{eff}}}\right) - \frac{32}{\pi^2}\sum_{k=1}^{\infty}\sum_{\substack{l=k+1\\ k+l\ \mathrm{odd}}}^{\infty}\frac{(-1)^k}{k^2-l^2} \\ &\quad \times\exp\left[-\frac{(k^2+l^2)\pi^2}{r_{\mathrm{eff}}}\right] + \mathrm{O}(L^{-2}). \end{aligned} \tag{47}$$

The rapid convergence to the scaling limit is shown in figure 2.

4. Discussion

We have studied DLC at the critical point where the particle density and other quantities display power laws in the thermodynamic limit. The problem can be made off-critical by introducing a birth process with rate Δ, corresponding to the back reaction of the coagulation process in (1).

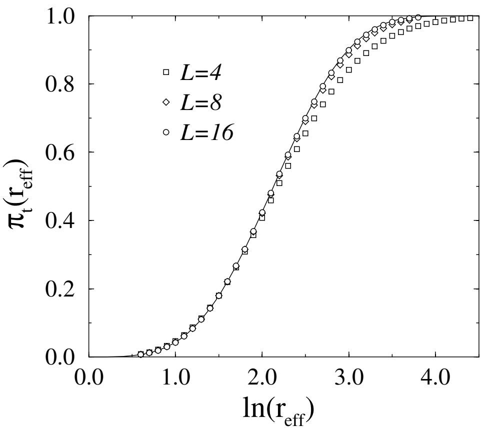

Figure 2. Variation of the scale-invariant crossing probability with the effective aspect ratio for free boundary conditions. The solid line corresponds to the asymptotic expression in equation (47).

The system being strongly anisotropic, under a change of the length scale by a factor b, the length transforms as $L' = L/b$ whereas the time transformation, $t' = t/b^z$, involves the anisotropy or dynamical exponent z [20]. For a scale-invariant crossing probability one obtains

$$P_t(L, t, \Delta) = P_t\left(\frac{L}{b}, \frac{t}{b^z}, b^{1/\nu}\Delta\right) \tag{48}$$

where ν is the exponent of the correlation length, $\xi = \hat{\xi}\Delta^{-\nu}$, which diverges at the critical point, $\Delta = 0$. Taking $b = \Delta^{-\nu}$ leads to

$$P_t(L, t, \Delta) = P_t\left(\frac{L}{\Delta^{-\nu}}, \frac{t}{\Delta^{-z\nu}}, 1\right) = f\left(\frac{L}{\xi}, \frac{t}{\tau}\right) \tag{49}$$

where we introduced the relaxation time, $\tau = \hat{\tau}\Delta^{-z\nu}$. Both the correlation length and the relaxation time contain a non-universal amplitude, $\hat{\xi}$ and $\hat{\tau}$, respectively. In equation (49), $f(x, y)$ is a universal scaling function of its dimensionless variables, $x = L/\xi$ and $y = t/\tau$. The finite-size scaling limit at criticality amounts to taking $\Delta = 0$ and $b = L$ in (48), which gives

$$P_t(L, t, 0) = P_t\left(1, \frac{t}{L^z}, 0\right) = \pi_t\left(c\frac{L^z}{t}\right). \tag{50}$$

Thus the crossing probability π_t is a scale-invariant universal function of the effective aspect ratio cr. The non-universal amplitude c depends on the choice of the length and time units. It can be expressed as a function of the non-universal correlation length and relaxation time amplitudes [21] by comparing (50) with (49). Since r appears through the dimensionless ratio x^z/y, one obtains

$$cr = \frac{(L/\xi)^z}{t/\tau} = \frac{\hat{\tau}}{\hat{\xi}^z}r \tag{51}$$

and $c = \hat{\tau}/\hat{\xi}^z$ which is equal to D^{-1} in our case.

We have shown with the example of DLC that the crossing probability π_t is a scale-invariant function of the effective aspect ratio for different types of boundary conditions. This function is expected to be universal as in the case of directed percolation [19]. An indication of the universality of π_t can be found in [35] where a birth process of the form $A \varnothing A \xrightarrow{2\lambda D} AAA$ was added to (1). The problem stays in the same universality class and remains exactly solvable through the empty interval method for all values of λ. It turns out that $I_n(t)$ is only modified by terms of higher order in $1/L$ which disappear in the finite-size scaling limit. One could also check the universality of π_t on the diffusion–annihilation problem, which has been shown to belong to the same universality class as DLC, through a similarity transformation in the quantum Hamiltonian formulation of the master equation [38–40]. Another possibility would be to verify the independence of the initial conditions by taking, for example, a site occupation probability smaller than the one in the initial state.

Finally, let us mention that a recent generalization of local scale invariance for strongly anisotropic systems [41, 42] leaves hope for directly obtaining the crossing probability formulae in a given universality class, as in isotropic systems.

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